

Bounds for the Convex Combination of the First Seiffert and Logarithmic Means in Terms of Generalized Heronian Mean

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Abstract

In this paper, bounds for convex combination of the first Seiffert and Logarithmic means by general Heronian mean are proved. The left bound is the best possible.

Keywords: Seiffert mean, Heronian mean, Logarithmic mean, convex combination, bounds

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1. Introduction

In the paper [1], authors proved the following optimal inequalities:
Let $a > 0$, $b > 0$, $a \neq b$ then

$$H_\delta(a, b) < P(a, b) < H_\beta(a, b) \quad \text{for } \delta \geq \pi - 2, \beta \leq 1 \quad (1)$$

and $\delta = \pi - 2$, $\beta = 1$ are the best constants.

$$H_\gamma(a, b) < L(a, b) < H_\tau(a, b) \quad \text{for } \gamma = +\infty, \tau \leq 4 \quad (2)$$

and $\gamma = +\infty$, $\tau = 4$ are the best constants.

$P(a, b)$ is the first Seiffert mean

$$P(a, b) = \frac{a - b}{4 \arctan\left(\sqrt{\frac{a}{b}}\right) - \pi} = \frac{a - b}{2 \arcsin\left(\frac{a-b}{a+b}\right)} \quad \text{for } a, b > 0, a \neq b. \quad (3)$$

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$L(a, b)$ is the Logarithmic mean

$$L(a, b) = \frac{a - b}{\log a - \log b} \quad \text{for } a, b > 0, a \neq b. \quad (4)$$

$H_\omega(a, b)$ is the Heronian mean

$$\begin{aligned} H_\omega(a, b) &= \frac{a + \omega\sqrt{ab} + b}{\omega + 2} \quad \text{for } -\infty < \omega < +\infty \\ &= \sqrt{ab} \quad \text{for } \omega = +\infty. \end{aligned} \quad (5)$$

It is well known, that $H_\omega(a, b)$ is a decreasing continuous function of the argument ω . From this and from results of [1] it follows that there exist optimal functions $p(\alpha)$, $q(\alpha)$, $0 \leq \alpha \leq 1$ such that

$$H_{p(\alpha)}(a, b) < \alpha P(a, b) + (1 - \alpha)L(a, b) < H_{q(\alpha)}(a, b).$$

The purpose of this paper is to establish suitable bounds for the optimal functions. For some details about means, see [1] and the related references.

2. Main Results

In this section we prove the following theorem.

Theorem 2.1. *Let $a, b > 0$, $a \neq b$. Let $P(a, b)$, $L(a, b)$, $H_\omega(a, b)$, be the first Seiffert mean, the Logarithmic mean and the Heronian mean. Let $\alpha \in [0, 1]$ and $p(\alpha)$, $q(\alpha)$, be the optimal functions such that: if for some p, q*

$$H_p(a, b) < \alpha P(a, b) + (1 - \alpha)L(a, b) < H_q(a, b) \quad (6)$$

then $p \geq p(\alpha)$ and $q \leq q(\alpha)$; then

$$p(\alpha) = \frac{\pi}{\alpha} - 2, \quad \frac{4}{1 + 3\alpha} < q(\alpha) < 4 - 3\alpha,$$

and $p(\alpha)$ is the best possible function.

Proof. (6) is equivalent to

$$\frac{1}{b}H_p(a, b) < \frac{\alpha}{b}P(a, b) + \frac{(1 - \alpha)}{b}L(a, b) < H_q(a, b). \quad (7)$$

Without loss of generality we can suppose that $0 < b < a$. Denote $t = \sqrt{b/a}$ and put

$$F(t, \alpha, \omega) = \alpha \frac{1 - t^2}{\pi - 4 \arctan t} + (1 - \alpha) \frac{1 - t^2}{-2 \ln t} - \frac{1}{\omega + 2} (t^2 + \omega t + 1). \quad (8)$$

Then $0 < t < 1$. First we prove the left inequality. This inequality is equivalent to $R(t, \alpha) = F(t, \alpha, \pi/\alpha - 2) > 0$ for $0 < t, \alpha < 1$. Some computation gives

$$R(t, \alpha) = \alpha \frac{1 - t^2}{\pi - 4 \arctan t} + (1 - \alpha) \frac{1 - t^2}{-2 \ln t} - \frac{\alpha t^2}{\pi} - \left(1 - \frac{2\alpha}{\pi}\right) t - \frac{\alpha}{\pi}. \quad (9)$$

Because of $R(t, \alpha)$ is a linear function of argument α , it suffices to show that $R(t, 0) > 0$ and $R(t, 1) > 0$ for $t \in (0, 1)$. Denote $F_1(t) = R(t, 0)$ and $F_2(t) = R(t, 1)$. Then

$$F_1(t) = \frac{s(t)}{2 \ln t} = \frac{t^2 - 1 - 2t \ln t}{2 \ln t}. \quad (10)$$

From $s''(t) = \frac{2t-2}{t} < 0$, $s'(1) = 0$ where $s'(t) = 2t - 2 \ln t - 2$ we have $s'(t) > 0$ and from $s(1) = 0$ we obtain $s(t) < 0$ and so $R(t, 0) > 0$. $F_2(t) = R(t, 1) > 0$ is equivalent to

$$h(t) = \pi - 4 \arctan t - \pi \frac{1 - t^2}{t^2 + (\pi - 2)t + 1} < 0. \quad (11)$$

Because of $h(0) = 0$, $h(1) = 0$ and $h'(0) = \pi^2 - 2\pi - 4 < 0$ where

$$h'(t) = -\frac{4}{1+t^2} - \pi \left(\frac{-2t(t^2 + (\pi - 2)t + 1) - (1 - t^2)(2t + \pi - 2)}{(t^2 + (\pi - 2)t + 1)^2} \right) \quad (12)$$

it suffices to show that the numerator of $h'(t)$ has only one root in $(0, 1)$. Some computation gives that $h'(t) = p(t)/((1+t^2)((t^2 + (\pi - 2)t + 1)^2))$ where

$$p(t) = (\pi^2 - 2\pi - 4)t^4 + (16 - 4\pi)t^3 + (-2\pi^2 + 12\pi - 24)t^2 + (16 - 4\pi)t + \pi^2 - 2\pi - 4. \quad (13)$$

It is easy to show that

$$p(t) = (1 - t^2)q(t) = (1 - t^2) \left((\pi^2 - 2\pi - 4)t^2 + (2\pi^2 - 8\pi + 8)t + \pi^2 - 2\pi - 4 \right). \quad (14)$$

From $q(0) = \pi^2 - 2\pi - 4 < 0$ and $q(1) = 4\pi(\pi - 3) > 0$ we have $R(t, 1) > 0$.

From

$$\lim_{t \rightarrow 0^+} F(t, \alpha, \omega) = \frac{\alpha}{\pi} - \frac{1}{\omega + 2}$$

we obtain that $p(\alpha)$ is the best possible function.

Now we prove the right bounds.

To show $q(\alpha) > 4/(4 + 3\alpha)$ we need to prove $F(t, \alpha, 4/(4 + 3\alpha)) < 0$ for all $t, \alpha \in (0, 1)$. This is equivalent to

$$G(t, \alpha) = \alpha \frac{1 - t^2}{\pi - 4 \arctan t} + (1 - \alpha) \frac{1 - t^2}{-2 \ln t} - \frac{1 + 3\alpha}{6 + 6\alpha} \left(t^2 + \frac{4}{1 + 3\alpha} t + 1 \right) < 0. \quad (15)$$

Some computation leads to

$$G''_{\alpha\alpha}(t, \alpha) = 6 \frac{1 - t^2}{(\pi - 4 \arctan t)(\ln t)} \{2 \ln t - 4 \arctan t + \pi\}. \quad (16)$$

Denote $u(t) = 2 \ln t - 4 \arctan t + \pi$. Because $u(1) = 0$ and

$$u'(t) = \frac{2t^2 - 4t + 2}{t(1 + t^2)} > 0$$

we have $u(t) < 0$ and $G''_{\alpha\alpha}(t, \alpha) > 0$. It implies $G(t, \alpha)$ is a convex function in argument α for each $t \in (0, 1)$. If we show that $G(t, 0) < 0$ and $G(t, 1) < 0$ for all $t \in (0, 1)$ then $G(t, \alpha) < 0$. $G(t, 0) < 0$ is equivalent to $s(t) = (3t^2 - 3)/(t^2 + 4t + 1) - \ln t > 0$. Simple computation leads to

$$s'(t) = \frac{ss(t)}{t(t^2 + 4t + 1)^2} = \frac{-t^4 + 4t^3 - 6t^2 + 4t - 1}{t(t^2 + 4t + 1)^2}.$$

From $ss'(t) = -4t^3 + 12t^2 - 12t + 4$, and $ss''(t) = -12t^2 + 24t - 12 < 0$, $ss'(1) = 0$, $ss(1) = 0$ we have $ss'(t) > 0$ and $ss(t) < 0$. From this and from $s(1) = 0$ we obtain $G(t, 0) < 0$.

$G(t, 1) < 0$ is equivalent to $v(t) = 3(t^2 - 1)/(t^2 + t + 1) - 4 \arctan t + \pi > 0$. From $v(1) = 0$ it suffices to show that $v'(t) < 0$. It follows from

$$v'(t) = \frac{ss(t)}{(1+t^2)(t^2+t+1)^2} = \frac{-t^4 + 4t^3 - 6t^2 + 4t - 1}{(1+t^2)(t^2+t+1)^2}.$$

To show $q(\alpha) < 4 - 3\alpha$ we need to prove $F(t, \alpha, 4 - 3\alpha) > 0$ for each α and some $t_\alpha \in (0, 1)$. Put $t_\alpha = 1 - \alpha + \alpha^2$. We show that $F(1 - \alpha + \alpha^2, \alpha, 4 - 3\alpha) > 0$ for $\alpha \in (0, 1)$. We use the following inequality

$$\frac{1-t^2}{-\ln t} > \frac{1-t^2}{\left(1-t + \frac{(1-t)^2}{2} + \frac{(1-t)^3}{3t}\right)}.$$

This inequality follows from the Taylor's series for $\ln t$, $t \in (0, 1)$

$$-\ln t < 1 - t + \frac{(1-t)^2}{2} + \frac{(1-t)^3}{3} + \frac{(1-t)^4}{3} + \dots$$

To prove $F(t, \alpha, 4 - 3\alpha) > 0$ for $t_\alpha = 1 - \alpha + \alpha^2$, $\alpha \in (0, 1)$ it suffices to show that

$$(6\alpha - 3\alpha^2) \frac{1 - t_\alpha^2}{\pi - 4 \arctan t_\alpha} + (3\alpha^2 - 9\alpha + 6) \frac{1 - t_\alpha^2}{2 \left(1 - t_\alpha + \frac{(1-t_\alpha)^2}{2} + \frac{(1-t_\alpha)^3}{3t_\alpha}\right)} \quad (17)$$

$$-t_\alpha^2 - (4 - 3\alpha)t_\alpha - 1 > 0.$$

Some calculation gives (17) is equivalent to

$$(6\alpha - 3\alpha^2) \frac{1 - t_\alpha^2}{\pi - 4 \arctan t_\alpha} > \frac{\varphi(t_\alpha)}{\psi(t_\alpha)}, \quad (18)$$

where

$$\varphi(t_\alpha) = t_\alpha^4 - (1 + 3\alpha)t_\alpha^3 + (9\alpha^2 - 12\alpha - 3)t_\alpha^2 + (9\alpha^2 - 21\alpha + 5)t_\alpha - 2, \quad (19)$$

$$\psi(t_\alpha) = t_\alpha^2 - 5t_\alpha - 2. \quad (20)$$

It is easy to show that

$$\begin{aligned} t_\alpha^2 &= 1 - 2\alpha + 3\alpha^2 - 2\alpha^3 + \alpha^4, \\ t_\alpha^3 &= 1 - 3\alpha + 6\alpha^2 - 7\alpha^3 + 6\alpha^4 - 3\alpha^5 + \alpha^6, \\ t_\alpha^4 &= 1 - 4\alpha + 10\alpha^2 - 16\alpha^3 + 19\alpha^4 - 16\alpha^5 + 10\alpha^6 - 4\alpha^7 + \alpha^8 \end{aligned}$$

$$\varphi(t_\alpha) = \alpha\varphi\varphi(\alpha) = \alpha(\alpha^7 - 7\alpha^6 + 27\alpha^5 - 61\alpha^4 + 91\alpha^3 - 105\alpha^2 + 72\alpha - 36). \quad (21)$$

It implies

$$\varphi\varphi(\alpha) < -36(1-\alpha)^2 - 46\alpha^2(1-\alpha)^2 + \alpha^2(-23 - 15\alpha^2 + 27\alpha^3 - 7\alpha^4 + \alpha^5). \quad (22)$$

Denote $k(\alpha) = -23 - 15\alpha^2 + 27\alpha^3 - 7\alpha^4 + \alpha^5$. We have $k(\alpha) < l(u) = -23 + 12u - 6u^2$, where $u = \alpha^2$. From $l(1) = -17$ and $l'(u) = 12(1-u) > 0$ we have $l(u) < 0$ so $\varphi(t_\alpha) < 0$. Evidently $\psi(t_\alpha) < 0$. Denote

$$H(\alpha) = 4 \arctan(t_\alpha) - \pi + (6\alpha - 3\alpha^2)(1 - t_\alpha^2) \frac{\psi(t_\alpha)}{\varphi(t_\alpha)}. \quad (23)$$

We need to show $H(\alpha) > 0$. From $H(1) = H(0) = 0$ it suffices to show that $H'(\alpha)$ has only one root in $(0, 1)$, and $H'(0) > 0$. Some computation gives

$$\begin{aligned} H(\alpha) &= 4 \arctan(1 - \alpha + \alpha^2) - \pi - \\ &\quad \frac{(6 - 3\alpha)(12\alpha - 24\alpha^2 + 25\alpha^3 - 14\alpha^4 - \alpha^5 + 5\alpha^6 - 4\alpha^7 + \alpha^8)}{-36 + 72\alpha - 105\alpha^2 + 91\alpha^3 - 61\alpha^4 + 27\alpha^5 - 7\alpha^6 + \alpha^7}. \end{aligned} \quad (24)$$

This can be rewritten as

$$H(\alpha) = 4 \arctan(1 - \alpha + \alpha^2) - \pi - \frac{a(\alpha)}{b(\alpha)}, \quad (25)$$

where

$$a(\alpha) = 72\alpha - 180\alpha^2 + 222\alpha^3 - 159\alpha^4 + 36\alpha^5 + 33\alpha^6 - 39\alpha^7 + 18\alpha^8 - 3\alpha^9, \quad (26)$$

$$b(\alpha) = -36 + 72\alpha - 105\alpha^2 + 91\alpha^3 - 61\alpha^4 + 27\alpha^5 - 7\alpha^6 + \alpha^7. \quad (27)$$

Some computation gives

$$H'(\alpha) = \frac{8\alpha - 4}{2 - 2\alpha + 3\alpha^2 - 2\alpha^3 + \alpha^4} - \frac{a'(\alpha)b(\alpha) - a(\alpha)b'(\alpha)}{b^2(\alpha)} = \quad (28)$$

$$\frac{v(\alpha)}{(2 - 2\alpha + 3\alpha^2 - 2\alpha^3 + \alpha^4)b(\alpha)^2}$$

where

$$v(\alpha) = \alpha^3(\alpha - 1)^3 s(\alpha) \quad (29)$$

$$s(\alpha) = -2304 + 11232\alpha - 26976\alpha^2 + 46372\alpha^3 - 59080\alpha^4 + 59711\alpha^5 - 48129\alpha^6 + 31498\alpha^7 - 16634\alpha^8 + 6980\alpha^9 - 2250\alpha^{10} + 513\alpha^{11} - 75\alpha^{12} + 6\alpha^{13}. \quad (30)$$

We used

$$b^2(\alpha) = 1296 - 5184\alpha + 12744\alpha^2 - 21672\alpha^3 + 28521\alpha^4 - 29838\alpha^5 + 25483\alpha^6 - 17852\alpha^7 + 10249\alpha^8 - 4778\alpha^9 + 1765\alpha^{10} - 500\alpha^{11} + 103\alpha^{12} - 14\alpha^{13} + \alpha^{14}, \quad (31)$$

$$a'(\alpha) = 72 - 360\alpha + 666\alpha^2 - 636\alpha^3 + 180\alpha^4 + 198\alpha^5 - 273\alpha^6 + 144\alpha^7 - 27\alpha^8, \quad (32)$$

$$b'(\alpha) = 72 - 210\alpha + 273\alpha^2 - 244\alpha^3 + 135\alpha^4 - 42\alpha^5 + 7\alpha^6, \quad (33)$$

$$a'(\alpha)b(\alpha) = -2592 + 18144\alpha - 57456\alpha^2 + 115200\alpha^3 - 159354\alpha^4 + 157122\alpha^5 - 103542\alpha^6 + 30120\alpha^7 + 24849\alpha^8 - 44007\alpha^9 + 36042\alpha^{10} - 19818\alpha^{11} + 7644\alpha^{12} - 2010\alpha^{13} + 333\alpha^{14} - 27\alpha^{15}, \quad (34)$$

$$a(\alpha)b'(\alpha) = 5184\alpha - 28080\alpha^2 + 73440\alpha^3 - 124776\alpha^4 + 150228\alpha^5 - 130083\alpha^6 + 76920\alpha^7 - 22338\alpha^8 - 9603\alpha^9 + 16890\alpha^{10} - 11610\alpha^{11} + 5031\alpha^{12} - 1434\alpha^{13} + 252\alpha^{14} - 21\alpha^{15}. \quad (35)$$

Now we show that $H'(\alpha)$ has only one root in $(0, 1)$ which is equivalent to $s(\alpha)$ has only one root in $(0, 1)$. Because $s(0) = -2304$ and $s(1) = 864$ it suffices to show that $s'(\alpha) > 0$

$$s'(\alpha) = 11232 - 53952\alpha + 139116\alpha^2 - 236320\alpha^3 + 298555\alpha^4 - 288774\alpha^5 + 220486\alpha^6 - 133072\alpha^7 + 62820\alpha^8 - 22500\alpha^9 + 5643\alpha^{10} - 900\alpha^{11} + 78\alpha^{12}. \quad (36)$$

Using $\alpha < 1$ we get

$$s'(\alpha) > ss(\alpha) = 11232 - 53952\alpha + 139116\alpha^2 - 236320\alpha^3 + 298555\alpha^4 - 288774\alpha^5 + 220486\alpha^6 - 133072\alpha^7 + 62820\alpha^8 - 22500\alpha^9 + 4743\alpha^{10}. \quad (37)$$

$$\frac{ss(\alpha)}{10^5} > p(\alpha) = 0.11 - 0.54\alpha + 1.39\alpha^2 - 2.364\alpha^3 + 2.985\alpha^4 - 2.888\alpha^5 + 2.204\alpha^6 \quad (38)$$

$$-1.331\alpha^7 + 0.628\alpha^8 - 0.23\alpha^9 + 0.04\alpha^{10}.$$

We show $p(\alpha) > 0$. We distinguish three cases.

Case 1. $\alpha \in (0, 0.45]$.

Case 2. $\alpha \in [0.45, 0.57]$.

Case 3. $\alpha \in [0.57, 1)$.

Case 1. Denote

$$p_1(\alpha) = 2.204\alpha^6 - 1.331\alpha^7 + 0.628\alpha^8 - 0.23\alpha^9 + 0.04\alpha^{10}. \quad (39)$$

$$p_2(\alpha) = 0.11 - 0.54\alpha + 1.39\alpha^2 - 2.364\alpha^3 + 2.985\alpha^4 - 2.888\alpha^5. \quad (40)$$

We have

$$p_1(\alpha) = \alpha^6 q_1(\alpha) = \alpha^6 (2.204 - 1.331\alpha + 0.628\alpha^2 - 0.23\alpha^3 + 0.04\alpha^4). \quad (41)$$

Some computations give

$$q_1'(\alpha) = -1.331 + 1.256\alpha - 0.69\alpha^2 + 0.16\alpha^3, \quad q_1''(\alpha) = 1.256 - 1.38\alpha + 0.48\alpha^2,$$

$$q_1'''(\alpha) = -1.38\alpha + 0.96\alpha.$$

From $q_1'''(\alpha) < 0$ and $q_1''(1) = 0.356$ we have $q_1''(\alpha) > 0$. From this and from $q_1'(1) = -0.605$ we obtain $q_1'(\alpha) < 0$. Because $q_1(1) = 1.311$ we have $q_1(\alpha) > 0$ and so $p_1(\alpha) > 0$ for $\alpha \in (0, 1)$.

$$p_2'(\alpha) = -0.54 + 2.78\alpha - 7.092\alpha^2 + 11.94\alpha^3 - 14.44\alpha^4,$$

$$p_2''(\alpha) = 2.78 - 14.184\alpha + 35.82\alpha^2 - 57.76\alpha^3.$$

Using the Cardano's formula we obtain that $p_2''(\alpha) = 0$ only for one real $\alpha^* = 0.3218153$. From $p_2'(0) = -0.54$, $p_2'(\alpha^*) = -0.1367703$, $p_2'(0.45) = -0.2292278$ we get $p_2'(\alpha) < 0$ for $\alpha \in (0, 0.45]$. From $p_2(0.45) = 0.0021674$ we have $p(\alpha) = p_1(\alpha) + p_2(\alpha) > 0$ for $\alpha \in (0, 0.45]$.

Case 2. Put

$$q_1(\alpha) = p_1(\alpha) - 0.77\alpha^5, \quad q_2(\alpha) = p_2(\alpha) + 0.77\alpha^5 \quad (42)$$

We have

$$q_1(\alpha) = \alpha^5 r_1(\alpha) = \alpha^5 (-0.77 + 2.204\alpha - 1.331\alpha^2 + 0.628\alpha^3 - 0.23\alpha^4 + 0.04\alpha^5). \quad (43)$$

Some computations give

$$r_1'(\alpha) = 2.204 - 2.662\alpha + 1.884\alpha^2 - 0.92\alpha^3 + 0.2\alpha^4,$$

$$r_1''(\alpha) = -2.662 + 3.768\alpha - 2.76\alpha^2 + 0.8\alpha^3,$$

$$r_1'''(\alpha) = 3.768 - 5.52\alpha + 2.4\alpha^2, \quad r_1''''(\alpha) = -5.52 + 4.8\alpha < 0.$$

From $r_1''''(\alpha) < 0$ and $r_1'''(0.57) = 1.40136$ we have $r_1'''(\alpha) > 0$. From this and from $r_1''(0.57) = -1.2628096$ we obtain $r_1''(\alpha) < 0$. From $r_1'(0.57) = 1.149506$ we have $r_1'(\alpha) > 0$ for $\alpha \in (0.45, 0.57]$. From $r_1(0.45) = 0.0008057$ we have $q_1(\alpha) > 0$ for $\alpha \in (0.45, 0.57]$.

$$q_2'(\alpha) = -0.54 + 2.78\alpha - 7.092\alpha^2 + 11.94\alpha^3 - 10.59\alpha^4,$$

$$q_2''(\alpha) = 2.78 - 14.184\alpha + 35.82\alpha^2 - 42.36\alpha^3,$$

$$q_2'''(\alpha) = -14.184 + 71.64\alpha - 127.08\alpha^2 \quad q_2''''(\alpha) = 71.64 - 254.16\alpha < 0.$$

From $q_2'''(0.45) = -7.6797$ we have $q_2'''(\alpha) < 0$. From $q_2''(0.45) = -0.209305$ we have $q_2''(\alpha) < 0$. From $q_2'(0.45) = -0.0713537$ we get $q_2'(\alpha) < 0$. From $q_2(0.57) = 0.003673$ we have $p(\alpha) = q_1(\alpha) + q_2(\alpha) > 0$ for $\alpha \in [0.45, 0.57]$.

Case 3. Using elementary computations we obtain

$$p(1) = 0.004, \quad p'(1) = -0.091, \quad p''(1) = -0.918, \quad p'''(1) = -6.966,$$

$$p^{(4)}(1) = -38.4, \quad p^{(5)}(1) = -161.64, \quad p^{(6)}(1) = -323.28,$$

$$p^{(7)}(1) = 1073.52, \quad p^{(8)}(1) = 14434.56, \quad p^{(9)}(1) = 61689.6, \quad p^{(10)}(1) = 145152.$$

From this we obtain

$$\begin{aligned} p(\alpha) = pp(1-\alpha) = & 0.004 + 0.091(1-\alpha) - 0.459(1-\alpha)^2 + 1.161(1-\alpha)^3 - 1.6(1-\alpha)^4 + \\ & 1.347(1-\alpha)^5 - 0.449(1-\alpha)^6 - 0.213(1-\alpha)^7 + 0.358(1-\alpha)^8 - 0.17(1-\alpha)^9 + 0.04(1-\alpha)^{10}. \end{aligned} \quad (44)$$

Denote $s_\alpha = 1 - \alpha$. If we show $pp(s_\alpha) > 0$ for $0 < s_\alpha < 0.44$ then the proof will be completed. Denote

$$pp_1(s_\alpha) = 0.004 - 0.449s_\alpha^6 - 0.213s_\alpha^7 + 0.358s_\alpha^8 - 0.17s_\alpha^9 + 0.04s_\alpha^{10}. \quad (45)$$

$$pp_2(s_\alpha) = 0.091s_\alpha - 0.459s_\alpha^2 + 1.161s_\alpha^3 - 1.6s_\alpha^4 + 1.347s_\alpha^5. \quad (46)$$

We have

$$pp_1'(s_\alpha) = s_\alpha^5 rr_1(s_\alpha) = s_\alpha^5 (-2.694 - 1.491s_\alpha + 2.864s_\alpha^2 - 1.53s_\alpha^3 + 0.4s_\alpha^4). \quad (47)$$

Some computations give

$$rr_1'(s_\alpha) = -1.491 + 5.728s_\alpha - 4.59s_\alpha^2 + 1.6s_\alpha^3.$$

Using the Cardano's formula we obtain that $rr_1''(s_\alpha) = 0$ only for one real $s_\alpha^* = 0.3435535$. From $rr_1(0) = -2.694$, $rr_1(0.44) = -2.9109087$, $rr_1(s_\alpha^*) = -2.9246712$ we get $pp_1'(s_\alpha) < 0$. From $pp_1(0.44) = 0.0004706$ we have $pp_1(s_\alpha) > 0$ for $s_\alpha \in (0, 0.44]$. Next we have

$$pp_2(s_\alpha) = s_\alpha z(s_\alpha) = s_\alpha (0.091 - 0.459s_\alpha + 1.161s_\alpha^2 - 1.6s_\alpha^3 + 1.347s_\alpha^4),$$

$$z'(s_\alpha) = -0.459 + 2.322s_\alpha - 4.8s_\alpha^2 + 5.388s_\alpha^3.$$

Using the Cardano's formula we obtain that $z'(s_\alpha) = 0$ only for one real $s_\alpha^{**} = 0.3534804$. From $z(0) = 0.091$, $z(0.44) = 0.280021$, $z(s_\alpha^{**}) = 0.0241802$ we have $pp_2(s_\alpha) > 0$ for $s_\alpha \in (0, 0.44]$. The proof is complete.

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